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DISCUSSION OF LATERAL BUCKLING OF ECCENTRICALLY LOADED I-SECTION COLUMNS

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By Jack R. Benjamin, Jacob Karol, and H. N. Hill and J. W. Clark

STRUCTURAL DIVISION

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DISCUSSION

Jack R. Benjamin,[17] Assoc. M. ASCE.—The study of lateral buckling (torsional failure) of I-sections is greatly assisted by the paper. However, the writer notes some serious shortcomings. Generally, if a theoretical check of experimental results is to be made, the theory must agree with the experimental conditions. The theoretical results quoted do not agree with the experiment performed. It is not sufficient to apply arbitrary factors of effective length and eccentricity without examining the implications of these factors.

The column under test will rotate torsionally under all loads. This rotation modifies the usual concepts. The rotation of column sections modifies the effective length. The point of zero moment about the minor axis is above the quarter point because of the rotation producing a moment which equals the major axis moment times the angular rotation.

The theoretical formulas quoted are not based on constant moment along the length of the member. They contain the influence of deflection on moment. The theory requires the use of the end moments at the ends of the effective length. These moments are experimentally applied and include the influence of a deflection. Thus the eccentricities in Table 2 should include this item if a check of the theory is desired.

The check between theory and experiment is remarkable considering the assumptions made. The theoretical solution of the experimental column can be made numerically without excessive work by determining the required equilibrium conditions at sections along the column length and then examining the compatible distortions.

The profession of structural engineering is greatly in need of some simple, rational, approximate theory of torsional failure. The presentation of empirical interaction formulas is not sufficient. Design requires an understanding of structural action much more than the ability to turn the crank on a formula. The writer hopes that the authors' studies may evolve some simple, rational, approximate theory.

Jacob Karol,[18] M. ASCE.—A valuable addition to the literature on lateral buckling is made by this paper. It is doubly interesting because the particular test data cited agree very well with the modified general theory of J. N. Goodier, and the theory explains why the previously suggested solutions (Eqs. 4 and 12) do not fit the test data. The authors state that they have been unable to express the solution in a form simple enough for design purposes. In the ensuing discussion a chart will be presented that the writer believes gives a solution conveniently and accurately.

A minor change in the modified general theory is proposed to permit the use of this theory for I-beams whose proportions may vary considerably from those tested by the authors. A more accurate expression for the critical value

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Note.—This paper by H. N. Hill and J. W. Clark was published in September, 1950, as *Proceedings-Separate No. 34*. The numbering of footnotes, illustrations, and equations in this Separate is a continuation of the consecutive numbering used in the original paper.

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of the bending moment (M_{cr}) is

$$M_{cr} = \frac{M'}{\sqrt{1 - \frac{I_y}{I_x}}} \qquad (13)$$

in which M' is the value given by Eq. 6.

The effect of the deflections in the plane of the web on the moments at the critical section can be determined by the following formula. The relation of the maximum moment ($M_{\max}$) to the applied moment (M) is given by

$$\frac{1 - \alpha}{1 + 0.272\,\alpha} = \frac{M}{M_{\max}} \qquad (14)$$

The average moment over the length of the specimen is given by

$$M_{av} = M + 0.636\,(M_{\max} - M) = \frac{M\,(1 - 0.191\,\alpha)}{1 - \alpha} \qquad (15)$$

Mr. Goodier's general theory as modified by the authors and the writer may then be written

$$\frac{P}{P'} + \left(\frac{M_{av}}{M_{cr}}\right)^2 = 1 - \beta\,\frac{P}{P'}\left(1 - \frac{P}{P'}\right) \qquad (16)$$

in which

$$\beta = \left(\frac{\rho\,P'}{M_{cr}}\right)^2 \qquad (17)$$

An important fact concerning Eq. 17 is that β is nondimensional. Its value is determined from Eqs. 5 and 13:

$$\beta = \frac{\left(\frac{2\,\rho_x}{d'}\right)^2\left[1 - \left(\frac{I_y}{I_x}\right)^2\right]}{1 + \frac{2\,J}{\pi^2\,(1 + \mu)\,I}\left(\frac{K\,L}{d'}\right)^2} \qquad (18)$$

The numerator of the right hand side cannot exceed one. The denominator will vary from a minimum of one for a short deep beam with wide flanges to a large number for a long shallow beam with narrow flanges. The value of the parameter β approaches one in the first case, and Eq. 16 reduces to a modification of Eq. 12. In the second case, the value of the parameter approaches zero, and Eq. 16 reduces to a modification of Eq. 4. Since the values of the parameter for the test specimens lie between 0.33 and 0.56, it is obvious that Eqs. 4 and 12 are not applicable. It should be noted that, since P' and M_{cr} must be calculated in the solution of a particular problem, Eq. 17 is more convenient for the calculation of β than Eq. 18.

Eqs. 15 and 16 have been plotted in Fig. 9. Two examples of the use of the chart as applied to the test data in Table 2 follow:

1. For specimen 13. Given $P' = 1{,}655$ lb, $M_{cr} = 4{,}479 \times 1.015 = 4{,}546$ in.-lb, $\rho = 1.588$ in., and $KL = 52.04$ in., find the maximum theoretical load for 1-in. eccentricity. Compute the Euler load in the plane of the web, $P_e =$

14,700 lb, and compute $\beta = 0.334$. The solution must be made indirectly. Estimate that $P = 1{,}396$ lb, for which $\alpha = P/P_e = 0.095$, and $P/P' = 0.845$. For these values, the chart shows $M/M_{av} = 0.92$ and $\frac{M_{av}}{M_{cr}} = 0.333$. Hence $M = 0.92 \times 0.333 \times 4{,}546 = 1{,}396$ in.-lb, and $e = 1$ in. The actual test load was 1,435 lb so the error equals -2.7%.

2. For specimen 7. Given $P' = 4{,}910$ lb, $M_{cr} = 10{,}949 \times 1.015 = 11{,}110$ in.-lb, $\rho = 1.592$ in., and $KL = 30.28$ in., find the maximum theoretical load

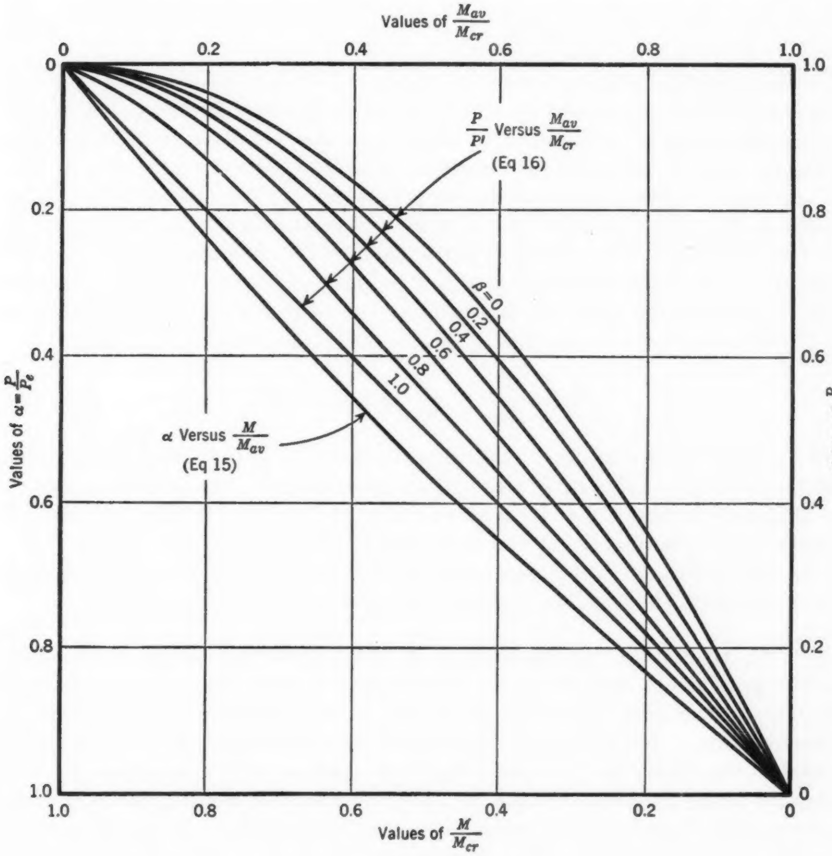

Fig. 9

for 1-in. eccentricity. Compute $P_e = 40{,}700$ lb and $\beta = 0.495$. Assume that $P = 3{,}810$ lb, hence $\alpha = 0.094$, and $P/P' = 0.776$. For these values the chart indicates that $M/M_{av} = 0.923$ and $M_{av}/M_{cr} = 0.372$. Hence $M = 0.923 \times 0.372 \times 11{,}110 = 3{,}810$ in.-lb and $e = 1$ in. The test load was 3,645 lb, giving an error of $+4.5\%$.

It is thus evident that use of the chart gives results substantially in agreement with those obtained by the authors. Because of the nondimensional

parameters, the chart is generally valid for the analysis of any I-beam eccentrically loaded in the plane of the web. For short beams where the critical stresses will be in the inelastic range, the writer suggests the use of the tangent modulus in solving Eqs. 5 and 13. It should be remembered, however, that in calculating the term α the Euler load is computed using the initial modulus.

Fig. 9 is an interaction (or stress-ratio) curve similar to those curves generally used in aircraft stress analysis. An equation of the working stress type can also be written as

$$\frac{M_c}{I} = \frac{1 - \alpha}{1 - 0.191\,\alpha} f_a \sqrt{\left(1 - \frac{P/A}{f'_c}\right)\left(1 - \beta\,\frac{P/A}{f'_c}\right)} \quad \text{.........(19)}$$

The danger in using this expression is that α must be computed for the actual working load (P) multiplied by the factor of safety used in the specifications, a point that may be overlooked by some designers. Furthermore, the factor of safety must be the same for bending and compressive stresses for Eq. 19 to be applicable. Assuming that the proper values are used, solution of a particular problem by Fig. 9 or Eq. 19 becomes a matter of individual preference.

The use of Fig. 9 in solving a particular problem is quite simple. Once a value of P has been assumed, α and P/P' are calculated directly. For the value of α, read the value M/M_{av} directly from the chart. For the values of P/P' and β, read the value of M_{av}/M_{cr} directly from the chart. Then

$$M = \frac{M_{av}}{M_{cr}} \times \frac{M}{M_{av}} \times M_{cr} = M \quad \text{.................(20)}$$

and $e = M/P$. If this checks the required value of e, the solution is correct. Otherwise, a new value of P must be assumed and the calculations repeated.

It should be noted that for a given value of P in a particular problem, the value of e is given directly by the procedure outlined above.

If the additional tests mentioned by the authors have been completed, it is hoped that they will be presented in the closing discussion.

H. N. Hill,[19] M. ASCE, and J. W. Clark,[20] Jun. ASCE.—Mr. Benjamin's statements concerning the differences between the experimental conditions and the assumptions on which the quoted theory is based are open to some question. The primary difference between theoretical and test conditions is that in the theory the bending moment about the major axis is assumed to be constant along the length of the member, whereas the bending moment in an eccentrically loaded column varies because of the deflection in the plane of the applied bending moment. Mr. Benjamin's suggestion advocating the use of the bending moment at the points of inflection rather than at the ends would be one way of taking approximate account of this effect for cases in which the ends of the member are fixed in the lateral direction.

Eqs. 9 and 10 may be derived on the basis of Mr. Goodier's equations of equilibrium, assuming the ends of the member to be either hinged, or fixed

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against lateral bending and rotation. The effect of the torsional rotation on the bending moment about the minor-axis is included in the derivation.

As Mr. Karol points out, an average bending moment, such as that defined by Eq. 15, may be substituted for M in Eqs. 9 and 10. In further analysis of the tests reported in this paper and subsequent test data[21] the writers have used an average moment defined by

$$M_{av} = \frac{M}{1 - \alpha} \qquad (21)$$

This expression gives satisfactory agreement between measured and calculated critical loads and is somewhat simpler than Eq. 15. In view of this agreement the writers suggest that a design formula for columns that are subject to eccentric loading and that fail by lateral buckling be based on the equation

$$\frac{M}{M'} = (1 - \alpha)\left(1 - \frac{P}{P'}\right) \qquad (22)$$

[21] "Lateral Buckling of Eccentrically Loaded I- and H-Section Columns," by H. N. Hill and J. W. Clark, *Proceedings*, U. S. National Congress of Applied Mechanics, June, 1951 (publication pending).